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VARIATIONAL PRINCIPLES FOR PERFECT GAS FLOWS WITH STRONG DISCONTINUITIES EXPRESSED IN EULER VARIABLES*

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Variational principles yielding equations of flow and conditions at strong discontinuities are considered in the case of a two-parameter gas in a strong potential force field. Functionals which are subjected to variations are expressed in terms of Euler variables and do not contain supplementary differential relations.

Similar principles were earlier constructed and applied primarily to continuous flows (see, e.g., /1—12/). Only few attempts were made in gasdynamics /13,14/ and related fields of continuous medium mechanics /15,16/ to apply them in the case of strong discontinuities. The Hamilton principle was extended in Lagrangian variables to unsteady discontinuous flows of perfect gas by Zamplen /17,18/. Similar principles for substantially more complex media were formulated by Sedov and his disciples (see /19,20/ and publications cited there) in the domain of Newtonian mechanics and of the theory of relativity (special and general).

For steady flows the starting point of investigations based on the use of the two-dimensional stream function appeared in /14/. The continuity of that function at shock waves made it possible to obtain relations at these in the form of Weirstrass—Erdman conditions. The vector potential lacks that property, and its application in /14/ only let to the formulation of the variational principle for continuous isoenergetic and isotropic three-dimensional streams, for which one of the simpler formulations of Batemen's principle provides the same result. Before the appearance of /14/, stream functions were used in the formulation of variational principles for continuous and discontinuous flows in /3—5/ and /13/, respectively. The varied functional of the form used in /14/ was also introduced in /3/.

Here, unlike in /14/, two stream functions are used as in /4/ for an arbitrary steady three-dimensional stream. The continuity of these functions at shock waves leads to the required conditions. At tangential discontinuities these functions usually become discontinuous, but the validity of the obtained principle is not violated, if the varied and the initial discontinuities are formed by the same streamlines. Note that Euler coordinates and particle displacements used in /17—20/ for formulating variational principles in Lagrangian variables have the same properties.

A modification of the Bateman's principle formulation mentioned above is proposed besides the principle that hold for any steady streams for piecewise isentropic and isoenergetic flows with tangential discontinuities of arbitrary intensity. This modification makes Bateman's principle valid also for this case. A similar modification is extended to three-dimensional unsteady streams with contact discontinuities. It is shown that the original formulation of Bateman's principle and its modification remain valid in the presence of discontinuities that admit isentropic investigation. It is pointed out that the principles proposed in /3,4/ for continuous steady isoenergetic flows in the absence of external forces, are in fact valid also in the presence of discontinuities. These principles are based on the use of one/3/ or two /4/ stream functions, which makes possible the extension to discontinuous solutions. The assumption in /4/ is isoenergy narrows the application domain of these principles and, also, complicates the analysis.

1. Let us recall the properties of the two stream functions ψ_1 and ψ_2 that are introduced below in the analysis of steady three-dimensional flows. Let ρ be the density and $\mathbf{q}$ the velocity vector. Then, as shown in /4/, the stream density vector $\rho\mathbf{q}$ can be represented in the form

$$\rho\mathbf{q} = \nabla\psi_1 \times \nabla\psi_2 \tag{1.1}$$

which satisfies the continuity equation $\nabla(\rho\mathbf{q}) = 0$. The stream functions ψ_i in (1.1) are scalar functions of coordinates. By virtue of equalities

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$$\mathbf{q}\nabla\psi_i \equiv (\nabla\psi_1 \times \nabla\psi_2)\nabla\psi_i = 0, \ i = 1, 2 \tag{1.2}$$

they are constant along each streamline, hence the latter may be considered as intersection lines of surfaces $\psi_1 = \text{const}$ and $\psi_2 = \text{const}$, for whose determination we use (1.2) after having specified in conformity with (1.1) a set of lines $\psi_i = \text{const}$ on some surface intersected by streamlines. This surface can, for instance, be taken in the unperturbed oncoming stream, where at least one of the set of lines (say $\psi_1 = \text{const}$) is constructed entirely arbitrarily. The stipulation of continuity of ψ_i at shock waves ensures the conservation of the stream mass over such discontinuities, since

$$\rho q_n \equiv \rho \mathbf{q}\mathbf{n} = \rho\mathbf{q}(\boldsymbol{\tau}_1 \times \boldsymbol{\tau}_2) = (\nabla\psi_1 \times \nabla\psi_2)(\boldsymbol{\tau}_1 \times \boldsymbol{\tau}_2) = \\ \psi_{11}\psi_{22} - \psi_{12}\psi_{21} \equiv D(\psi_{ij}) \ (\psi_{ij} = \partial\psi_i/\partial\tau_j) \tag{1.3}$$

where $\boldsymbol{\tau}_1, \boldsymbol{\tau}_2, \mathbf{n}$ is the right-hand set of three mutually orthogonal unit vectors, with vector $\mathbf{n}$ of the normal to the discontinuity surface, and q_n is the projection of $\mathbf{q}$ on $\mathbf{n}$. Since the derivatives of ψ_{ij} are determined in directions tangent to the jump, the continuity of ψ_i ensures the continuity of ρq_n. At tangential discontinuity surfaces and along impenetrable walls $D \sim q_n = 0$, which, incidentally, is automatically satisfied, if on such surface (more exactly, on some of its "initial" line intersected by streamlines) one of the stream functions is assumed constant.

In a steady stream of perfect gas with specific enthalpy i in the absence of external forces the total enthalpy $I = i + q^2/2$ is constant along each streamline independent of the presence of discontinuities. Hence $I = I(\psi_1, \psi_2)$, where the continuous or discontinuous function $I(\psi_1, \psi_2)$ is determined by conditions in the oncoming stream. In the presence of an external mass force $\mathbf{F} = -\nabla U$, where the potential energy U is a known function of coordinates, $J(\psi_1, \psi_2) = I + U$ plays the part of I. Unlike I and J the specific entropy s increases at transition through compression shocks. Hence $s = S^k(\psi_1, \psi_2)$, with functions $S^k(\psi_1, \psi_2)$ proper for each subregion V^k in which space V occupied by the stream is divided by shock waves.

2. We introduce the functional

$$L = \iiint\limits_V \rho\left(\frac{q^2}{2} - e - U + J\right) dV \tag{2.1}$$

in which the internal energy $e = i - p/\rho = e(\rho, s)$ is a known function of ρ and s. Since $Tds = de + pd(1/\rho)$, where T is the absolute temperature and p the pressure, hence

$$e_\rho \equiv (\partial e/\partial\rho)_s = p/\rho^2, \quad e_s \equiv (\partial e/\partial s)_\rho = T \tag{2.2}$$

We rewrite (2.1) using (1.1) and the properties of J and s, in the form

$$L = \iiint\limits_V \left\{\frac{|\nabla\psi_1 \times \nabla\psi_2|^2}{2\rho} - \rho e(\rho, S^k(\psi_1, \psi_2)) - \rho U + \rho J(\psi_1, \psi_2)\right\} dV \tag{2.3}$$

and shall show that the conditions of steadiness with respect to variations ρ, ψ_i and displacements Δn of strong discontinuity surfaces $\partial V_d = \partial V_s \bigcup \partial V_c$, where $\partial V_s (\partial V_c)$ are compression shock surfaces (tangential discontinuities), yield the equation of flow and conditions of conservation at discontinuities, if in each subregion V^k, S^k is assumed to be a fixed function of its arguments (see the end of this paper).

Then, for any arbitrary function φ under variation $\delta\varphi$ we take the difference between the varied and unvaried values for fixed independent variables. The displacement Δn is measured along the normal $\mathbf{n}$ to ∂V_d. Parameters on different sides of ∂V_d are denoted by indices plus and minus and the remainder $\varphi_+ - \varphi_-$ is denoted by $[\varphi]$. Besides $\delta\varphi$ we introduce on each side of ∂V_d the increment $\Delta\varphi$ which we define as the difference of φ at points of varied and unvaried discontinuities lying on the same normal. It can be shown (see, e.g., /21/) that on each side of ∂V_d

$$(\Delta\varphi)_\pm = \{\delta\varphi + (\partial\varphi/\partial n)_{\tau_1, \tau_2}\Delta n\}_\pm = \delta\varphi_\pm + (\mathbf{n}\nabla\varphi_\pm)\Delta n \tag{2.4}$$

where $\mathbf{n}$ is either the unit vector of its normal ("external" to the considered side of ∂V_d) i.e. $\mathbf{n}_+$ or $\mathbf{n}_- = -\mathbf{n}_+$, or any of vectors $\mathbf{n}_+$ and $\mathbf{n}_-$. In either case the validity of (2.4) is ensured by the simultaneous change of signs of $\mathbf{n}$ and Δn.

Varying (2.3) together with (2.2) and (2.4) we take into account the following. The sequence of equalities

$$(2\rho)^{-1}\delta \mid \nabla\psi_1 \times \nabla\psi_2|^2 = (2\rho)^{-1}\delta \{(\nabla\psi_1 \times \nabla\psi_2)(\nabla\psi_1 \times \nabla\psi_2)\} = -\mathbf{q}\{\nabla\psi_2 \times \nabla(\delta\psi_1)\} + \mathbf{q}\{\nabla\psi_1 \times \nabla(\delta\psi_2)\} = (\nabla\psi_2 \times \mathbf{q})\nabla(\delta\psi_1) - (\nabla\psi_1 \times \mathbf{q})\nabla(\delta\psi_2) = \{\nabla(\mathbf{q} \times \nabla\psi_2)\}\delta\psi_1 - \{\nabla(\mathbf{q} \times \nabla\psi_1)\}\delta\psi_2 - \nabla\{(\mathbf{q} \times \nabla\psi_2)\delta\psi_1 - (\mathbf{q} \times \nabla\psi_1)\delta\psi_2\} = (\boldsymbol{\omega}\nabla\psi_2)\delta\psi_1 - (\boldsymbol{\omega}\nabla\psi_1)\delta\psi_2 + \nabla\{(\mathbf{q} \times \nabla\psi_1)\delta\psi_2 - (\mathbf{q} \times \nabla\psi_2)\delta\psi_1\} \quad (\boldsymbol{\omega} = \nabla \times \mathbf{q}) \tag{2.5}$$

is valid.

The volume integral in δL of the last term of (2.5) is reduced in the usual manner to integrals on both sides of ∂V_d, body surfaces ∂V_b, and the remaining sections of boundary ∂V_e of the considered flow region. After this the integrands of surface integrals contain $(\mathbf{q} \times \nabla\psi_i)_n = (\mathbf{q} \times \nabla\psi_i)\mathbf{n}$. Since $\mathbf{n} = \boldsymbol{\tau}_1 \times \boldsymbol{\tau}_2$, hence

$$(\mathbf{q} \times \nabla\psi_i)_n = q_{\tau 1}\psi_{i2} - q_{\tau 2}\psi_{i1} \tag{2.6}$$

where $q_{\tau i}$ is the projection of $\mathbf{q}$ on $\boldsymbol{\tau}_i$. Since in turn $\boldsymbol{\tau}_1 = -\mathbf{n} \times \boldsymbol{\tau}_2$, and $\boldsymbol{\tau}_2 = \mathbf{n} \times \boldsymbol{\tau}_1$, it follows from (1.1) that

$$\rho q_{\tau 1} = \psi_{12}(\nabla\psi_2)_n - \psi_{22}(\nabla\psi_1)_n, \quad \rho q_{\tau 2} = \psi_{21}(\nabla\psi_1)_n - \psi_{11}(\nabla\psi_2)_n \tag{2.7}$$

Using (2.4), (2.6), and (2.7) we obtain that in the surface integrals in δL with respect to $\partial V = \partial V_d \cup \partial V_b \cup \partial V_e$

$$(\mathbf{q} \times \nabla\psi_1)_n\delta\psi_2 - (\mathbf{q} \times \nabla\psi_2)_n\delta\psi_1 = (q_{\tau 1}\psi_{12} - q_{\tau 2}\psi_{11})\Delta\psi_2 - (q_{\tau 1}\psi_{12} - q_{\tau 2}\psi_{22})\Delta\psi_1 - pq_\tau^2\Delta n \quad (q_\tau^2 = q_{\tau 1}^2 + q_{\tau 2}^2) \tag{2.8}$$

with $\Delta n = 0$ and $\Delta\psi_i = \delta\psi_i$ on $\partial V_b \cup \partial V_e$.

Using (2.2), (2.5), and (2.8), taking into account the total contribution in δL of displacement of discontinuities on Δn, and combining the integrals on different sides, we finally obtain

$$\delta L = \iiint\limits_V \Big\{\Big(J(\psi_1, \psi_2) - \frac{q^2}{2} - i - U\Big)\delta\rho + (\boldsymbol{\omega}\nabla\psi_2 - \rho T S_{\psi 1} + \rho J_{\psi 1})\delta\psi_1 - (\boldsymbol{\omega}\nabla\psi_1 + \rho T S_{\psi 2} - \rho J_{\psi 2})\delta\psi_2\Big\} dV + \iint\limits_{\partial V_d} \Big\{\Big[\rho\Big(\frac{q_n^2 - q_\tau^2}{2} - e - U + J\Big)\Big]\Delta n - [(q_{\tau 1}\psi_{22} - q_{\tau 2}\psi_{21})\Delta\psi_1] + [(q_{\tau 1}\psi_{12} - q_{\tau 2}\psi_{11})\Delta\psi_2]\Big\} d\Sigma - \iint\limits_{\partial V_b \cup \partial V_e} \{(q_{\tau 1}\psi_{22} - q_{\tau 2}\psi_{21})\delta\psi_1 - (q_{\tau 1}\psi_{12} - q_{\tau 2}\psi_{11})\delta\psi_2\} d\Sigma \tag{2.9}$$

where $S_{\psi i}$ and $J_{\psi i}$ are partial derivatives with respect to ψ_i, the superscript k at S is omitted, and $d\Sigma$ is an element of the respective surface.

The condition of steadiness of L with respect to ρ is provided by the energy integral

$$q^2 / 2 + i + U = J(\psi_1, \psi_2) \tag{2.10}$$

while the conditions of steadiness with respect to ψ_i lead to two scalar equations that are equivalent to the single vector equation

$$\mathbf{q} \times \boldsymbol{\omega} = \nabla J - T\nabla S \tag{2.11}$$

We stress that (2.11) indeed yields two, not three, scalar equations, since the projections of the left- and right-hand sides of this vector equation onto the streamline direction are identically zero.

Relations at shock waves are obtained in the form of steadiness conditions (2.9) with respect to Δn and $\Delta\psi_i$ on ∂V_s. Recalling that here ψ_i and consequently also $\Delta\psi_i$ are continuous, and equating the coefficients at $\Delta\psi_i = \Delta\psi_{i-} = \Delta\psi_{i+}$ to zero, we find that

$$[q_{\tau 1}]\psi_{22} - [q_{\tau 2}]\psi_{21} = 0, \quad [q_{\tau 1}]\psi_{12} - [q_{\tau 2}]\psi_{11} = 0$$

The determinant of coefficients at $[q_{\tau i}]$ which by virtue of (1.3) is equal ρq_n and nonzero. Hence the above equalities imply the continuity of components of $\mathbf{q}$ tangent to the shock

$$[q_{\tau 1}] = 0, \quad [q_{\tau 2}] = 0 \tag{2.12}$$

Then equating to zero the coefficients at Δn on ∂V_s and eliminating J in the obtained formula, using (2.10), we obtain the condition of conservation of the momentum component normal to the shock

$$[\rho q_n^2 + p] = 0 \tag{2.13}$$

The condition of conservation of total enthalpy at the shock follows directly from Eq. (2.10), if one writes it twice (ahead and behind the shock) and takes into account the continuity of U and J on the shock.

U can become discontinuous only in the presence of external surface forces (when J and not I is conserved on ∂V_s). The continuity of J on ∂V_s was assumed right from the beginning.

The condition of invariance of L on ∂V_c with respect to Δn also yields (2.13), after the elimination of J with the use of (2.10). But at a tangential discontinuity $q_n = 0$, hence also here

$$[p] = 0 \tag{2.14}$$

According to (1.3) when q_n is zero, the determinant $D(\psi_{ij})$ of coefficients at $q_{\tau i}$ in multipliers at $\Delta\psi_i$ or $\delta\psi_i$ in the integrals with respect to ∂V_c and ∂V_b in (2.9). This enables us to write (2.9) in the form

$$\delta L = \iint\limits_{\partial V_c} \left[(q_{\tau 1}\psi_{22} - q_{\tau 2}\psi_{21}) \left(\frac{\psi_{12}}{\psi_{22}} \Delta\psi_2 - \Delta\psi_1 \right) \right] d\Sigma + \iint\limits_{\partial V_b} (q_{\tau 1}\psi_{22} - q_{\tau 2}\psi_{21}) \left(\frac{\psi_{12}}{\psi_{22}} \delta\psi_2 - \delta\psi_1 \right) d\Sigma - \iint\limits_{\partial V_e} \{(q_{\tau 1}\psi_{22} - q_{\tau 2}\psi_{21})\,\delta\psi_1 - (q_{\tau 1}\psi_{12} - q_{\tau 2}\psi_{11})\,\delta\psi_2\}\, d\Sigma$$

At every point of ∂V_b and of both sides of ∂V_c which are stream surfaces, it is convenient to direct vector $\boldsymbol{\tau}_1$ along the velocity vector. Then $q_{\tau 1} = q$, $q_{\tau 2} = 0$ and the expression for δL becomes

$$\delta L = \iint\limits_{\partial V_c} [q(\psi_{12}\Delta\psi_2 - \psi_{22}\Delta\psi_1)]\, d\Sigma + \iint\limits_{\partial V_b} q(\psi_{12}\Delta\psi_2 - \psi_{22}\Delta\psi_1)\, d\Sigma - \iint\limits_{\partial V_e} \{(q_{\tau 1}\psi_{22} - q_{\tau 2}\psi_{21})\,\delta\psi_1 - (q_{\tau 1}\psi_{12} - q_{\tau 2}\psi_{11})\,\delta\psi_2\}\, d\Sigma \tag{2.15}$$

Let, moreover, the varied and original surfaces ∂V_c and ∂V_b be formed by streamlines that correspond to the same ψ_1 and ψ_2. Without going into a detailed discussion of this situation, we would point out that it is often automatically realized. For instance, it is so at tangential discontinuities that begin in the oncoming stream, on bodies in unperturbed streams with attached bow shocks and at tangential discontinities that are formed behind these, on "external" impenetrable surfaces in problems of flow in channels, etc. When the assumed condition is satisfied, $\Delta\psi_i$ on ∂V_c and $\delta\psi_i$ on ∂V_b are nonzero only becuase of the shift of streamlines on respective surfaces. By defining such shift by the quantity $\Delta\tau_2$, i.e. displacement of streamlines in the direction of vector $\boldsymbol{\tau}_2$ normal to it by virtue of the selection of $\boldsymbol{\tau}_1$, it is possible to show that $\Delta\psi_i = -\psi_{i2}\Delta\tau_2$ on ∂V_c and $\delta\psi_i = -\psi_{i2}\Delta\tau_2$ on ∂V_b. The use of these equalities reduces to zero the first two integrals in the right-hand side of (2.15).

Omitting a detailed investigation of the last integral in δL, we point out the following. In any specific problem ∂V_e is either absent or decomposed in surfaces bounding the considered flow on different sides. Thus it is sometimes justified to introduce a finite region on whose boundaries appropriate boundary conditions are established, instead of considering the infinite stream flow over bodies. Then, as a rule, the orientation of vector $\mathbf{q}$ is known in the oncoming stream (on ∂V_e^-) and reasonably far behind the body (on ∂V_e^+). The latter makes it possible to reduce the terms with $\delta\psi_i$ to zero by selecting $\partial V_e^\pm$ normal to $\mathbf{q}$. Then $q_{\tau i} = 0$ and the last integral in the right-hand side of (2.15) vanishes for any $\delta\psi_i$. Along sections ∂V_e that coincide with fixed stream surfaces, the same integrals vanish because here $\delta\psi_i = 0$.

Formulation of the variational principle for the three-dimensional case was based in /4/ on the analysis of functional

$$L = \iint\limits_{V} (\rho q^2 + p)\, dV \tag{2.16}$$

in which the integrand is, by virtue of the assumption of isoenergy, a function of ψ_i and

$\nabla\psi_i$. Although only continuous flows were investigated in /4/, the analysis similar to the described above shows that in the case of (2.16) the Weirstrass—Erdman conditions yield the same conditions at strong discontinuities as (2.1). Incidentally, we would point out that when $U = 0$ in the case of solutions corresponding to a real flow, i.e. which satisfy the energy integral (2.10), formula (2.1) reduces to (2.16). However the use of the variational principle in the form (2.1) is preferable not only owing to its greater generality but, also, because the analysis that assumes the independence of ρ proves to be simpler.

3. Let us now restrict the analysis to the narrower class of piecewise isoenergetic and piecewise isentropic steady streams with tangential discontinuities of any intensity and shock waves that are weak in the sense of entropy increase, and can be considered with a high degree of accuracy ($[s] \sim [p]^3$) is isentropic approximation. It is then possible to introduce the continuous potential φ^k such that $\mathbf{q} = \nabla\varphi^k$ in each subregion V^k into which, unlike previously, the stream is split only by tangential discontinuities, including those occurring in flows over bodies, but not by shock waves. Introduction in V^k of the continuous potential as opposed to the stream functions ψ_i, automatically ensures the continuity of velocity vector components $q_{\tau i} = \partial\varphi^k / \partial\tau_i$ tangent to shocks. The energy integral is, with allowance for φ^k written in the form

$$(\nabla\varphi^k)^2 / 2 + i\,(p, S^k) + U = J^k \tag{3.1}$$

where, unlike in Sect.2, S^k and J^k are constant in V^k. Moreover (3.1) ensures the continuity of total enthalpy at compression shocks. Additionally, it follows from this that p is a known function of $\nabla\varphi^k, U, S^k$, and J^k. However, since U is a fixed function of coordinates, and S^k and J^k are fixed constants, we shall write $p = p^k(\nabla\varphi^k)$. In conformity with the equality $Tds = di - (1/\rho)\,dp$ and (3.1) we have

$$\delta p = -(\rho\nabla\varphi^k)\,\delta\,(\nabla\varphi^k) \tag{3.2}$$

We introduce the functional

$$L = \iint\limits_V p^k(\nabla\varphi^k)\,dV \tag{3.3}$$

which differs the functional of one of the formulations of Bateman's principle only by the presence of superscripts. If (3.3) yields the variational principle, then by virtue of the above condition of invariance of L to variations φ^k and displacements of discontinuity surfaces must yield besides the continuity equation, the continuity of the mass stream and of the normal component of momentum on weak shocks (the continuity of I on these in the corollary of (3.1)), the condition of impenetrability on tangential discontinuities and body surfaces, and condition (2.14) on tangential discontinuities. We shall show that it is in fact so and that, consequently, (3.3) leads to the required principle. It is obvious that the validity of the used here formulation of Bateman's principle for isoenergetic and isentropic flows with tangential discontinuities and weak shock with, is simultaneously proved.

The calculation of δL is carried out as in Sect.2. Taking into account certain differences related to the use of (3.2), we finally obtain

$$\delta L = \iiint\limits_V \{\nabla(\rho\nabla\varphi)\}\,\delta\varphi\,dV + \iint\limits_{\partial V_d} \{[\rho(\nabla\varphi)_n^2 + p]\,\Delta n - [\rho(\nabla\varphi)_n\Delta\varphi]\}\,d\Sigma - \iint\limits_{\partial V_b \cup \partial V_e} \rho(\nabla\varphi)_n\,\delta\varphi\,d\Sigma \tag{3.4}$$

(here and subsequently the superscript k is omitted).

From this, equating to zero the coefficient at $\delta\varphi$ in the volume integral, we obtain the continuity equation

$$\nabla(\rho\nabla\varphi) = 0 \tag{3.5}$$

where the density is a known function of $\nabla\varphi$ for each subregion V^k. The form of the latter is defined by the integral (3.1) and the equation of state of the form $\rho = \rho(i, s)$.

The condition of impenetrability of bodies surfaces $(\nabla\varphi)_n = 0$ is obtained by equating to zero the coefficient at $\delta\varphi$. The same condition is obtained for both sides of each tangential discontinuity as the result of equating to zero of the coefficients at $\Delta\varphi_+$ and $\Delta\varphi_- \neq \Delta\varphi_+$ on ∂V_c. The condition of continuity of p on ∂V_c is then obtained by investigating the coefficient at Δn.

For shock waves where the potential φ is continuous and, consequently $\Delta\varphi_+ = \Delta\varphi_-$, we similarly obtain from (3.4)

$$[\rho(\nabla\varphi)_n] = 0, \quad [\rho(\nabla\varphi)_n^2 + p] = 0$$

which are the conditions necessary for completing the proof of the statement made above. The remaining in (3.4) integral over ∂V_e shows that in the approximation considered in the presence of such boundaries it is convenient to specify the constancy of φ on them or, as in Sect.2, take the stream surface as ∂V_e.

4. Let us consider unsteady flows with contact discontinuities of any intensity and weak shock waves that are the analog of steady flows in Sect.3. For these the part of (3.1) is played in each subregion V^k by the integral

$$\varphi_t^k + (\nabla\varphi^k)^2 / 2 + i\,(p, S^k) + U = J^k\,(t) \quad (\varphi_t = \partial\varphi / \partial t) \tag{4.1}$$

which implies that $p = p^k\,(\varphi_t^k, \nabla\varphi^k)$ in the same sense as in Sect.3, and

$$\delta p = -\rho\delta\varphi_t^k - (\rho\nabla\varphi^k)\,\delta\,(\nabla\varphi^k) = \{\rho_t + \nabla\,(\rho\nabla\varphi^k)\}\,\delta\varphi^k - (\rho\delta\varphi^k)_t - \nabla\,\{(\rho\nabla\varphi^k)\,\delta\varphi^k\} \tag{4.2}$$

is substituted for (3.2).

Having in mind the extension of Bateman's principle to the considered case we introduce the functional

$$L = \iiiint\limits_{\Omega = V\times\tau} p^k\,(\varphi_t^k, \nabla\varphi^k)\,d\Omega \tag{4.3}$$

in which τ is the time interval. We shall show that the condition of invariance of L to variation of φ and displacements of discontinuities provide all necessary equations and conditions. For this, using (4.2) and proceeding as in Sects.2 and 3 (taking, of course, into account singularities of the space of four variables), we variate (4.3). We obtain

$$\delta L = \iiiint\limits_{\Omega} \{\rho_t + \nabla\,(\rho\nabla\varphi)\}\,\delta\varphi\,d\Omega + \iiint\limits_{\partial\Omega_d} \{[\rho\delta\varphi]\,D - [\rho\,(\nabla\varphi)_n\,\delta\varphi] + [p]\,\Delta n\}\,d\Sigma\,dt + \iiint\limits_{\partial\Omega_b\cup\partial\Omega_e} \rho\,\{D - (\nabla\varphi)_n\}\,\delta\varphi\,d\Sigma\,dt$$

where (and subsequently the superscript k is omitted), $\partial\Omega_d = \partial V_d \times \tau$, D is the velocity of the respective surface along the normal to itself, the first term in the integral over $\partial\Omega_d$ is the result of integration with respect to t, and the second, the result of passing from the integral over V to that over ∂V. When determining δL, the variations $\delta\varphi$ at the time interval ends were disregarded. The obtained expressions for δL with allowance for (2.4) assume the form

$$\delta L = \iiiint\limits_{\Omega} \{\rho_t + \nabla\,(\rho\nabla\varphi)\}\,\delta\varphi\,d\Omega +$$
$$\iiint\limits_{\partial\Omega_d} \{[\rho\,(\nabla\varphi)_n\,((\nabla\varphi)_n - D) + p]\,\Delta n + [\rho\,(D - (\nabla\varphi)_n)\,\Delta\varphi]\}\,d\Sigma\,dt + \iiint\limits_{\partial\Omega_b} \rho\,(D - (\nabla\varphi)_n)\delta\varphi\,d\Sigma\,dt$$

(the integral over $\partial\Omega_e$ is subsequently omitted).

From this, as in Sect.3, we obtain the equations of continuity, the conditions of impenetrability $D - (\nabla\varphi)_n = 0$ at the body surfaces and contact discontinuities, the stipulation of pressure continuity on the latter, and the equalities

$$[\rho\,(D - (\nabla\varphi)_n)] = 0,\ [\rho\,(\nabla\varphi)_n\,((\nabla\varphi)_n - D) + p] = 0$$

on shock waves. Conservation of components $\mathbf{q} = \nabla\varphi$ tangent to shocks is implied, as in Sect. 3 by the continuity of potential on them, and the condition of total enthalpy conservation (in the shock system)

$$[(D - (\nabla\varphi)_n)^2 + 2i] = 0$$

is obtained as the corollary of (4.1) of continuity of φ, J, U and of that at the shock

$$[\varphi_t] + D\,[(\nabla\varphi)_n] = 0$$

Note, that, as shown by the preceding, the success of investigation of flows with strong discontinuities is linked with the selection of the function which remains continuous on shock waves. Moreover, in the presence of regions of continuous entropy change these functions must remain valid for a particle and ensure that the continuity equation is satisfied. Continuity of these functions at tangential discontinuities is not required. It seems that also in the general three-dimensional unsteady case it is expedient to use three "stream functions" (Lagrangian variables) when formulating variational principles. The introduction in the four-dimensional space $(t, \mathbf{x})$ of the "velocity vector" $\mathbf{q}^\circ$ with components $1, q_1, q_2, q_3$, where q_i are projections of $\mathbf{q}$ on the x_i-axes and of operator ∇° with components $\partial / \partial t$ and $\partial / \partial x_i$, then the "stream functions" ψ_i, which ensure the fulfillment of continuity equations and remain valid in the particle, can be defined by the equality $\rho \mathbf{q}^c = \nabla^\circ\psi_1 \times \nabla^\circ\psi_2 \times \nabla^\circ\psi_3$.

Finally, we shall point out one singularity of the variational principles expounded in /3,4,14/ and in Sect.2 which must be kept in mind when applying them to flows with shock waves. In a real flow the displacement of shock surfaces and the flow perturbation upstream of the latter affects functions $S^k(\psi_1, \psi_2)$, while in the derivation of conditions of steadiness they are assumed above to be fixed functions ψ_1 and ψ_2. This fact which was also noted in /14/ reduces the value of these principles and must be always taken into consideration in their application (for instance, in the construction of direct methods of solving problems of flow over bodies). This problem does not occur for any of the considered above variational principles in the case of weak shocks for which the isoentropic approximation is valid. We should point out in connection with this that the variational principles whose steadiness conditions yield among other things, relations at shock wave without, however, distinguishing between compression and rarefaction shocks. This makes it necessary to improve the respective principles by the inclusion the inhibition of entropy decrease. This relates also to principles that are formulated in Lagrangian coordinates.

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